Q 1. If $u(x,y)=1+x+y+f(xy)$, where is a differentiable function, then u satisfies-

(a) $x\frac{\partial u}{\partial x}-y\frac{\partial u}{\partial y}=x^2-y^2$ (b) $x\frac{\partial u}{\partial x}-y\frac{\partial u}{\partial y}=0$

(c) $x\frac{\partial u}{\partial x}-y\frac{\partial u}{\partial y}=x-y$ (d) $y\frac{\partial u}{\partial x}-x\frac{\partial u}{\partial y}=x-y$

Sol. $u=1+x+y+f(xy)$ ... (i)

Differentiate (i) partially with respect to x; we get

$$\frac{\partial u}{\partial x}=1+y\,f'(xy)\Rightarrow\frac{\partial u}{\partial x}-1=y\,f'(xy) \quad \text{... (ii)}$$

Differentiate (i) partially with respect to y; we get

$$\frac{\partial u}{\partial y}=1+x\,f'(xy)\Rightarrow\frac{\partial u}{\partial y}-1=x\,f'(xy) \quad \text{... (iii)}$$

Now solve equations (ii) and (iii)

$$\frac{\frac{\partial u}{\partial x}-1}{\frac{\partial u}{\partial y}-1}=\frac{y}{x}\Rightarrow x\frac{\partial u}{\partial x}-y\frac{\partial u}{\partial y}=x-y$$

Option (c) is correct answer

Q 2. If $z=f\left(xe^y\right)+g\left(y^2\cos y\right)$ where f and g are arbitrary functions then partial differential equation of minimum order is-

(a) $z_{xy}+xz_{xx}=z_x$ (b) $z_{xy}+xz_{xx}=xz_x$

(c) $z_{xy}-xz_{xx}=z_x$ (d) $z_{xy}-xz_{xx}=xz_x$

Sol. $z=f\left(xe^y\right)+g\left(y^2\cos y\right)$ ... (i)

Differentiate (i) partially with respect to x; we get

$z_x = e^y f'(xe^y)$ ... (ii)

Differentiate (ii) partially with respect to x and y respectively; we get

$z_{xx} = e^{2y} f''(xe^y)$ ... (iii) and $z_{xy} = xe^{2y} f''(xe^y) + e^y f'(xe^y)$ ... (iv)

Using (ii), (iii) and (iv); we get

$z_{xy} = xz_{xx} + z_x \Rightarrow z_{xy} - x z_{xx} = z_x$ Option (c) is correct answer

Q 3. Consider the Lagrange equation $x^2 \frac{\partial z}{\partial x} + y^2 \frac{\partial z}{\partial y} = (x+y)z$

Then the general solution of the given equation is

1. $F\left(\frac{xy}{z}, \frac{x-y}{z}\right) = 0$ for an arbitrary differentiable function F

2. $F\left(\frac{x-y}{z}, \frac{1}{x} - \frac{1}{y}\right) = 0$ for an arbitrary differentiable function F

3. $z = f\left(\frac{1}{x} - \frac{1}{y}\right)$ for an arbitrary differentiable function f

4. $z = xy\, f\left(\frac{1}{x} - \frac{1}{y}\right)$ for an arbitrary differentiable function f

Sol. Given $x^2 \frac{\partial z}{\partial x} + y^2 \frac{\partial z}{\partial y} = (x+y)z$ ... (i)

Lagrange's Auxiliary equation is

$\frac{dx}{x^2} = \frac{dy}{y^2} = \frac{dz}{(x+y)z}$ ... (ii)

Using multipliers $\langle 1, -1, 0 \rangle$ and $\left\langle \frac{1}{x}, \frac{1}{y}, -\frac{1}{z} \right\rangle$ in (ii); we get

$$\frac{dx}{x^2} = \frac{dy}{y^2} = \frac{dz}{(x+y)z} = \frac{dx-dy}{x^2-y^2} = \frac{\frac{1}{x}dx + \frac{1}{y}dy - \frac{1}{z}dz}{0}$$

From **1**st and **2**nd fractions

$\frac{dx}{x^2} = \frac{dy}{y^2} \Rightarrow \frac{1}{x} - \frac{1}{y} = c_1$ ... (iii)

From **3**rd and **4**th fractions

$$\frac{dz}{(x+y)z} = \frac{dx-dy}{x^2-y^2} \Rightarrow \frac{dz}{z} = \frac{dx-dy}{x-y}$$

$$\Rightarrow \log(x-y)=\log z+\log c_2 \quad \Rightarrow \frac{x-y}{z}=c_2 \qquad \text{... (iv)}$$

From last fraction

$$\frac{1}{x}dx+\frac{1}{y}dy-\frac{1}{z}dz=0 \Rightarrow \log x+\log y-\log z=\log c'$$

$$\Rightarrow \frac{xy}{z}=c_3 \qquad \text{... (v)} \qquad \text{or} \qquad \frac{z}{xy}=c_4 \qquad \text{... (vi)}$$

From equations (iv) and (v), the general solution is

$F\left(\frac{xy}{z},\frac{x-y}{z}\right)=0$ So option 1st is correct.

From equations (iii) and (iv), the general solution is

$F\left(\frac{x-y}{z},\frac{1}{x}-\frac{1}{y}\right)=0$ So option 2nd is correct.

From equations (iii) and (vi), the general solution is

$\frac{z}{xy}=f\left(\frac{1}{x}-\frac{1}{y}\right) \Rightarrow z=xyf\left(\frac{1}{x}-\frac{1}{y}\right)$ So option 4th is correct.

but option 3 is incorrect as there doesn't exist any fraction which gives $z=c_5$

So options 1, 2 and 4 are correct

Shortcut: -

Lagrange Auxiliary equation is

$$\frac{dx}{x^2}=\frac{dy}{y^2}=\frac{dz}{(x+y)z} \qquad \text{... (i)}$$

From 1st and 2nd fractions

$$\frac{dx}{x^2}=\frac{dy}{y^2} \quad \Rightarrow \frac{1}{x}-\frac{1}{y}=c_1 \qquad \text{... (ii)}$$

Now consider $\frac{x-y}{z}=c_2$

Then $\frac{z(dx-dy)-(x-y)dz}{z^2}=0 \Rightarrow z(dx-dy)-(x-y)dz=0 \Rightarrow \frac{dx-dy}{x-y}=\frac{dz}{z}$

... (iii)

Equation (iii) can be obtained by taking multipliers $\langle 1,-1,0\rangle$ in (i)

$\Rightarrow \frac{x-y}{z}=c_2$ is solution of given Lagrange equation

Now consider $\frac{xy}{z} = c_3$

Then $\frac{z(xdy + ydx) - xydz}{z^2} = 0 \Rightarrow z(xdy + ydx) - xydz = 0 \Rightarrow \frac{xdy + ydx}{xy} = \frac{dz}{z}$ ... (iv)

Equation (iv) can be obtained by taking multipliers $\langle y, x, 0 \rangle$ in (i)

$\Rightarrow \frac{xy}{z} = c_3$ is solution of given Lagrange equation

Now consider $z = c_4$

$\Rightarrow dz = 0$ which can't be obtained from any fraction

Therefore solution obtained by (i), (ii) and (iv) are general solutions i.e. option 1, 2 and 4 are correct.

Q 4. The integral solution of partial differential equation $z_x + z_y = z$ passing through $x = t$, $y = 2t$, $z = 1$ is

(a) e^{x+2y} (b) e^{2x-y}

(c) e^{2x+y} (d) e^{y-2x}

Sol. Given partial differential equation is $z_x + z_y = z$

Lagrange's Auxiliary equation is

$\frac{dx}{1} = \frac{dy}{1} = \frac{dz}{z}$ ... (i)

From 1st and 2nd fractions; we get

$x - y = c_1$ ... (ii)

From 1st and 3rd fractions; we get

$x = \log z + \log c' \Rightarrow ze^{-x} = c_2$ ... (iii)

The general solution is given by

$ze^{-x} = \phi(x - y)$ ... (iv)

Now put $x = t$, $y = 2t$, $z = 1$ in (iv), we get

$e^{-t} = \phi(t - 2t) \Rightarrow \phi(-t) = e^{-t} \Rightarrow \phi(x - y) = e^{x-y}$

Therefore equation (iv) becomes

$ze^{-x} = e^{x-y} \Rightarrow z = e^{2x-y}$

Q 5. Let $u(x,y)$ be the solution of $x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=4u$ satisfying the condition $u(x,y)=1$ on the circle $x^2+y^2=1$. Then $u(2,2)$ equals______

Sol. Given partial differential equation is $x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=4u$

Lagrange's Auxiliary equation is

$$\frac{dx}{x}=\frac{dy}{y}=\frac{dz}{4z} \quad \text{... (i)}$$

From 1st and 2nd fractions; we get

$$\frac{x}{y}=c_1 \quad \text{... (ii)}$$

From 2nd and 3rd fractions; we get

$$\frac{y^4}{z}=c_2 \quad \text{... (iii)}$$

The general solution is given by

$$\frac{y^4}{z}=\phi\left(\frac{x}{y}\right) \quad \text{... (iv)}$$

Put $x=\cos\theta$, $y=\sin\theta$, $z=1$ in equation (iv); we get

$$\frac{\sin^4\theta}{1}=\phi\left(\frac{\cos\theta}{\sin\theta}\right)\Rightarrow \sin^4\theta=\phi(\cot\theta)\Rightarrow \frac{1}{\left(1+\cot^2\theta\right)^2}=\phi(\cot\theta)$$

$\Rightarrow \phi\left(\frac{x}{y}\right)=\frac{y^4}{\left(x^2+y^2\right)^2}$. Therefore equation (iv) becomes

$$\frac{y^4}{z}=\frac{y^4}{\left(x^2+y^2\right)^2}\Rightarrow z=\left(x^2+y^2\right)^2\Rightarrow z(2,2)=(4+4)^2=64$$

Q 6. For the Cauchy problem $u_t-uu_x=0$, $x\in\mathbb{R}$, $t>0$ with $u(x,0)=x$, which of the following statements is/are true?

(a) The solution u exists for all $t>0$

(b) The solution u exists for $t<\frac{1}{2}$ and breaks down at $t=\frac{1}{2}$

(c) The solution u exists for $t<1$ and breaks down at $t=1$

(d) The solution u exists for $t<2$ and breaks down at $t=2$

Sol. Given partial differential equation is $u_t-uu_x=0$

Lagrange's Auxiliary equation is

$$\frac{dx}{-u}=\frac{dt}{1}=\frac{du}{0} \qquad \text{... (i)}$$

From 3rd fraction; we get

$$u=c_1 \qquad \text{... (ii)}$$

Use equation (ii) in equation (i); we get

$$\frac{dx}{-c_1}=\frac{dt}{1}\Rightarrow x+c_1t=c_2 \quad \Rightarrow x+ut=c_2 \qquad \text{... (iii)}$$

Given $u(x,0)=x$

Put $x=s$, $t=0$, $u=s$ in equations (ii) and (iii); we get

$$s=c_1 \text{ and } s=c_2\Rightarrow c_1=c_2\Rightarrow u=x+ut \;\Rightarrow u=\frac{x}{1-t}$$

Therefore the solution exists for $t\neq 1$ and breaks down at $t=1$. So option (c) is correct.

Q 7. The solution of the partial differential equation $u_t-xu_x+1-u=0$, $x\in\mathbb{R}$, $t>0$ subject to $u(x,0)=g(x)$ is

(a) $u(x,t)=1-e^{t}\left(1-g\left(xe^{t}\right)\right)$ (b) $u(x,t)=1+e^{t}\left(1-g\left(xe^{t}\right)\right)$

(c) $u(x,t)=1-e^{-t}\left(1-g\left(xe^{-t}\right)\right)$ (d) $u(x,t)=e^{-t}\left(1-g\left(xe^{t}\right)\right)$

Sol. Given partial differential equation is $u_t-xu_x+1-u=0\Rightarrow u_t-xu_x=u-1$

Lagrange's Auxiliary equation is

$$\frac{dx}{-x}=\frac{dt}{1}=\frac{du}{u-1} \qquad \text{... (i)}$$

From 1st and 2nd fractions, we get

$$\log x=-t+\log c_1\Rightarrow x=e^{-t}c_1 \;\Rightarrow xe^{t}=c_1 \qquad \text{... (ii)}$$

From 2nd and 3rd fractions, we get

$$t=\log(u-1)+\log c_2\Rightarrow t=\log(u-1)c_2 \;\Rightarrow e^{t}=(u-1)c_2\Rightarrow \frac{e^{t}}{u-1}=c_2 \qquad \text{... (iii)}$$

Given $u(x,0)=g(x)$

Put $x=s$, $t=0$, $u=g(s)$ in equations (i) and (ii); we get

$$s=c_1 \text{ and } \frac{1}{g(s)-1}=c_2\Rightarrow \frac{1}{g(c_1)-1}=c_2\Rightarrow \frac{1}{g\left(xe^{t}\right)-1}=\frac{e^{t}}{u-1}$$

$\Rightarrow u-1=e^t\left(g\left(xe^t\right)-1\right) \Rightarrow u(x,t)=1-e^t\left(1-g\left(xe^t\right)\right)$. So option (a) is correct.

Q 8. Consider the partial differential equation $x\frac{\partial u}{\partial x}+yu\frac{\partial u}{\partial y}=-xy$ for $x>0$ subject to $u=5$ on $xy=1$ then

(a) $u(x,y)$ exists when $xy \le 19$ and $u(x,y)=u(y,x)$ for $x>0$, $y>0$

(b) $u(x,y)$ exists when $xy \ge 19$ and $u(x,y)=u(y,x)$ for $x>0$, $y>0$

(c) $u(1,11)=3,\ u(13,-1)=7$

(d) $u(1,-1)=5,\ u(11,1)=-5$

Sol. Given partial differential equation is $x\frac{\partial u}{\partial x}+yu\frac{\partial u}{\partial y}=-xy$

Lagrange's Auxiliary equation is

$\frac{dx}{x}=\frac{dy}{yu}=\frac{du}{-xy}=\frac{ydx+xdy+udu}{xy}$ Using multipliers $\langle y,x,u\rangle$

From 3rd and 4th fractions; we get

$$xy+\frac{u^2}{2}+u=c_1 \qquad \text{... (i)}$$

Given $u=5$ on $xy=1$

Put $xy=1$ and $u=5$ in equation (i), we get

$$1+\frac{25}{2}+5=c_1 \Rightarrow c_1=\frac{37}{2}$$

Put the value of c_1 in equation (i)

$$xy+\frac{u^2}{2}+u=\frac{37}{2} \Rightarrow u^2+2u+2xy=37 \Rightarrow (u+1)^2=38-2xy \Rightarrow u=-1\pm\sqrt{38-2xy}$$

The solution u exists for $38-2xy \ge 0 \Rightarrow xy \le 19$

Also $u(1,11)=3,-5$, $u(13,-1)=7,-9$ and $u(1,-1)=-1\pm 2\sqrt{10}$

Therefore option (a) and (c) are correct.

Q 9. The solution of the initial value problem

$(x-y)\frac{\partial u}{\partial x}+(y-x-u)\frac{\partial u}{\partial y}=u$ with $u(x,0)=1$ satisfies

(a) $u^2(x-y+u)+(y-x-u)=0$

(b) $u^2(x+y+u)+(y-x-u)=0$

(c) $u^2(x-y+u)-(x+y+u)=0$

(d) $u^2(y-x+u)+(x+y-u)=0$

Sol. Given partial differential equation is $(x-y)\frac{\partial u}{\partial x}+(y-x-u)\frac{\partial u}{\partial y}=u$

Lagrange's Auxiliary equation is

$$\frac{dx}{x-y}=\frac{dy}{y-x-u}=\frac{du}{u}=\frac{dx+dy+du}{0} \qquad \text{Using multipliers } \langle 1,1,1\rangle$$

From last fraction; we get

$$x+y+u=c_1 \quad \text{... (i)}$$

Using (i) in 2nd fraction, we get

$$\frac{dy}{2y-c_1}=\frac{du}{u}\Rightarrow\frac{\log(2y-c_1)}{2}=\log u+\log c_2\Rightarrow c_2{}^2u^2=2y-c_1\Rightarrow\frac{y-x-u}{u^2}=c_2{}^2 \quad \text{... (ii)}$$

Given $u(x,0)=1$

Put $x=t$, $y=0$, $u=1$ in (i) and (ii); we get

$c_1=1+t$ and $c_2{}^2=-(1+t)\Rightarrow c_2{}^2+c_1=0\Rightarrow\frac{y-x-u}{u^2}+x+y+u=0$

$\Rightarrow u^2(x+y+u)+(y-x-u)=0$.

So option (b) is correct.

Q 10. Consider the partial differential equation $z_x-z_y=2$ with initial data curve $\Gamma:(t,t,2t)$. Then number of solutions of given problem is

(a) One (b) Zero

(c) Two (d) Infinite Solutions

Sol. Compare the given partial differential equation with $P(x,y,z)p+Q(x,y,z)q=R(x,y,z)$; we have

$P(x,y,z)=1,\ Q(x,y,z)=-1,\ R(x,y,z)=2$

Put $x=t$, $y=t$, $z=2t\Rightarrow x'(t)=1,\ y'(t)=1,\ z'(t)=2$

Now $\frac{x'(t)}{P(x(t),y(t),z(t))}=1,\ \frac{y'(t)}{Q(x(t),y(t),z(t))}=-1,\ \frac{z'(t)}{R(x(t),y(t),z(t))}=1$

Since $\frac{x'(t)}{P(x(t),y(t),z(t))}\neq\frac{y'(t)}{Q(x(t),y(t),z(t))}$. Therefore given partial differential equation has unique solution.